Selfdual strings and loop space Nahm equations

Andreas Gustavsson

Förstamajgatan 24, S-415 10, Göteborg, Sweden
E-mail: `a.r.gustavsson@swipnet.se`

ABSTRACT: We give two independent arguments why the classical membrane fields should be take values in a loop algebra. The first argument comes from how we may construct selfdual strings in the M5 brane from a loop space version of the Nahm equations. The second argument is that there appears to be no infinite set of finite-dimensional Lie algebras (such as $su(N)$ for any N) that satisfies the algebraic structure of the membrane theory.

Contents

1. Introduction

This paper concerns the selfdual string soliton in the M5 brane. We assume a straight string with SO(4) rotational symmetry, and we wish to analyze its shape for SU(2) gauge group proceeding by analogy with the magnetic monopole construction in super Yang-Mills theory.

Having strings with SO(4) rotational symmetry, it is very natural to expect that the equation that corresponds to the Nahm equation in the construction of magnetic monopoles, should also have this SO(4) symmetry. Moreover, it is natural to suspect that a fuzzy three-sphere would play a similar role for the selfdual string construction as the fuzzy two-sphere plays for the magnetic monopole.

More precisely, for monopoles one has solutions to the Nahm equation of the form

$$T^I \sim \frac{t^I}{z-1} \tag{1.1}$$

near $z = 1$, where t^I are coordinates on a fuzzy two-sphere. That is, t^I are some representation matrices for SU(2). For the selfdual string the analog of this would be

$$T^i \sim \frac{G^i}{\sqrt{z-1}} \tag{1.2}$$

near $z = 1$, where G^i now are coordinates on a fuzzy three-sphere. This approach was taken in [7, 3] among others.

The construction of the fuzzy three-sphere [2] is complicated compared to the fuzzy two-sphere. But if one notices the isomorphism $su(2) \oplus su(2) \simeq so(4)$, one could think that the fuzzy three sphere could be described in terms of (or at least be mapped to) two fuzzy two-spheres.

If G^i denote the coordinates on a fuzzy three-sphere in a certain reducible representation of SO(4), characterized by the single integer n (see [2] for details), and if we define $G^{ij} = \frac{1}{2}[G^i, G^j]$, and let λ^I_{ij} denote the 't Hooft matrices (defined in appendix A), then

$$t^I = \frac{1}{2}\lambda^I_{ij}G^{ij}$$
$$\tilde{t}^I = \frac{1}{2}\tilde{\lambda}^I_{ij}G^{ij} \tag{1.3}$$

can be computed, for instance from eq. (17) in [3].[1] The result is very simple,

$$t^I = \sum_r \rho_r(\sigma^I)$$
$$\tilde{t}^I = \sum_r \rho_r(\tilde{\sigma}^I) \tag{1.7}$$

where $[\sigma^I, \sigma^J] = 2i\epsilon_{IJK}\sigma^K$ are the Pauli sigma matrices. There is a tensor product and $\rho_r(\sigma^I)$ means $1 \otimes \cdots \otimes \sigma^I \otimes \cdots \otimes 1$ where σ^I is placed at position $r = 1, \ldots, n$. This shows that there is a map from the fuzzy three-sphere coordinates to the coordinates t^I and $\tilde{t}^I$ of two fuzzy two-spheres,

$$4it^I = \epsilon_{IJK}[t^J, t^K]$$
$$4i\tilde{t}^I = \epsilon_{IJK}[\tilde{t}^J, \tilde{t}^K] \tag{1.8}$$

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[1]Eq. (17) in [3] reads

$$G^{ij}\mathcal{P}_\pm = \sum_{r=1}^n \left(\frac{n+3}{4}\rho_r(\gamma^{ij}P_\pm) - \frac{n+1}{4}\rho_r(\gamma^{ij}P_\mp) \right) \mathcal{P}_\pm \tag{1.4}$$

(For $n = 1$, this is $G^{ij} = \gamma^{ij}$). From this we get

$$\lambda^I_{ij}G^{ij}\mathcal{P}_+ = (n+3)\sum_r \rho_r(\sigma^I)$$
$$\lambda^I_{ij}G^{ij}\mathcal{P}_- = -(n+1)\sum_r \rho_r(\sigma^I). \tag{1.5}$$

Here $4\sigma^I = \lambda^I_{ij}\gamma^{ij}$. Then the sum $\lambda^I_{ij}G^{ij} \equiv 2t^I$ becomes

$$t^I = \sum_r \rho_r(\sigma^I). \tag{1.6}$$

This suggests that it might be possible to describe the fuzzy three-sphere in this alternative way as two fuzzy two-spheres.

One could now also suspect that the selfdual string can be viewed in some sense as two decoupled sets of magnetic monopoles. This is what we wish to make precise in this paper.

We will work in loop space, and use loop space fields, that is, fields that depend not on a point but on a whole loop C embedded in space. In particular we have a loop space gauge field $A_{\mu s}(C)$ which, in the abelian case, may be given the ultra local form

$$A_{\mu s}(C) = B_{\mu\nu}(C(s))\dot{C}^\nu(s) \tag{1.9}$$

where B denotes the usual two-form gauge field, which is one of the fields in the tensor multiplet in six dimensions.

In [10] it was shown that $(2,0)$ supersymmetry can be extended to non-abelian loop space fields under quite general assumptions. One need not assume ultra locality for this to work. Neither do we need any assumptions on ultra locality in what we do in this paper. We wish to leave ultra locality as an open crucial question. Eventually, if one wants to make concrete computations, one will have to face the question. For the time being we content ourselves with setting up some algebraic framework in loop space, that should put some constraints on what $(2,0)$ theory could be. Loop space is a very big space, and it seems to us like $(2,0)$ theory could live somewhere in it. But we do not know exactly where it lives. More precisely, one should find the appropriate constraints on the loop space fields.

In the abelian case, we notice that we need the three-form field strength $H_{\mu\nu\rho}$ to be anti-self-dual in order to close two supersymmetry variations into a superalgebra on the equations of motion. However, as was seen in [10], we do not need to enforce self-duality on the corresponding loop space field strength $F_{\mu s,\nu t}$ in order to close supersymmetry on-shell. Intuitively one can understand why this should be so, by inserting the condition of self-duality into the loop space field strength. We then find that

$$\begin{aligned} F_{\mu s,\nu t}(C) &= H_{\mu\nu\rho}(C(s))\dot{C}^\rho(s)\delta(s-t) \\ &= \frac{1}{6}\epsilon^{\kappa\tau\sigma}{}_{\mu\nu\rho}H_{\kappa\tau\sigma}(C(s))\dot{C}^\rho(s)\delta(s-t) \end{aligned} \tag{1.10}$$

but the second line in this equation can not be expressed in terms of $F_{\mu s,\nu t}(C)$, which shows that we can not implement self-duality on $F_{\mu s,\nu t}(C)$.[2] Happily, in [10] we could show that no 'self-duality' on $F_{\mu s,\nu t}$ is needed in order to close supersymmetry on-shell. Self-duality comes out from the loop space supersymmetry formalism once we insert the ultra local equation eq. (1.9) for the loop space fields into the loop space supersymmetry variations.

In [11, 12] a gerbe formalism was developed with ultra local expressions for non-abelian loop space gauge fields as defined in [13] based on ideas in [14]. This formalism uses a local

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[2]Possibly one may relate $F(C)$ to some sort of hodgedual of $F(C')$ when evaluated at a different loop C' so that one would have a non-local self-duality condition in loop space. Now supersymmetry acts locally in loop space and relates the variation of a bosonic (fermionic) field at C to fermionic (bosonic) fields evaluated at the same point C, hence any non-local constraints on loop space fields do not concern us when we study supersymmetry.

connection two-form $B_{\mu\nu}$ as well as a local connection one-form A_μ subject to vanishing fake curvature.[3] Vanishing fake curvature was shown in [11, 12] is needed in order for Wilson surfaces to be well-defined. Presumably there are corresponding ultra local expressions for the other loop space fields in the tensor multiplet as well. It may be that one just need to hang on a gauge index on the scalars and fermions. Inspired by how things work in the abelian case, one could try and play the same game with such non-abelian ultra local loop space fields. Plugging such loop space fields into the supersymmetry variations in loop space obtained in [10], one can derive corresponding equations for the local fields (for the $B_{\mu\nu}$ and A_μ). From such a computation one should find a non-abelian counterpart of the self-duality condition on $B_{\mu\nu}$. But now, just like self-duality could not be imposed in loop space, vanishing fake curvature can probably not either. But again, just like self-duality comes out from supersymmetry when we assume ultra locality in the abelian case, in the non-abelian case the condition of vanishing fake curvature could also come out from supersymmetry. Though this is just a speculative idea.

It would of course be extremely nice if an ultra local approach could be made to work consistently with supersymmetry, since then one could derive equations of motion for local space-time fields from loop space equations, and possibly also start to do real physics computations in non-abelian $(2,0)$ theory.

Either way, we think that any possible set of equations that contains $(2,0)$ theory one way or the other, should be worth to study, even if it may be difficult to tell at the moment what these equations could be used for. Eventually $(2,0)$ theory (if eventually properly understood) could be useful in describing non-perturbative effects in QCD [15], as well as giving a better understanding of M-theory and quantum gravity via AdS-CFT correspondence.

The redaction is as follows: In section 2 we introduce suitable coordinates in loop space, that generalizes the Hopf map $S^3 \mapsto S^2$. In section 3 we derive the Bogomolnyi equation for selfdual strings in terms of these new loop space coordinates. In section 3.1 we present an abelian solution and a loop space generalization of the ADHMN construction of selfdual strings. In section 4 we show how the corresponding Nahm equations can be obtained from the membrane theory. In section 4.1 we show that this membrane theory can be reduced to super Yang-Mills theory. In section 5 we show the relation between two Nahm equations and the Basu-Harvey equation in Ref [7].

2. Various coordinates in loop space

We assume static straight strings in $1 + 5$ dimensions. Hence we have translational invariance in $1 + 1$ dimensions, and we will restrict our attention to the transverse space to the strings, $\mathbb{R}^4$, with cartesian coordinates x^i. We then also consider the loop space over $\mathbb{R}^4$. If we parametrize the embedding of a loop in $\mathbb{R}^4$ as $s \mapsto C^i(s)$, we have coordinates for this loop that are $C^i(s)$ with s being a continuous index. Hence loop space is an infinite-dimensional space.

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[3]Something like $F_{\mu\nu} \sim B_{\mu\nu}$ where F is the curvature of A, though generically F and A can take values in different internal algebras and one must map B to the same algebra as A to make sense of this equation.

If the loop is planar, then we can compute the area that it encloses by integrating

$$\sigma_s^{ij} := C^i(s)\dot{C}^j(s) - C^j(s)\dot{C}^i(s) \tag{2.1}$$

around the loop. But we now define this quantity for any loop, and then we may define a new set of coordinates on loop space as[4]

$$\begin{aligned} X_s^I &:= \frac{1}{2}\lambda_{ij}^I\sigma_s^{ij} \\ \tilde{X}_s^I &:= \frac{1}{2}\tilde{\lambda}_{ij}^I\sigma_s^{ij} \end{aligned} \tag{2.2}$$

that we may invert to get

$$\sigma_s^{ij} = -\frac{1}{2}\left(\lambda_{ij}^I X_s^I + \tilde{\lambda}_{ij}^I \tilde{X}_s^I\right) \tag{2.3}$$

We now claim that either X or $\tilde{X}$ can be used as coordinates on loop space. To see this we would like to invert the maps

$$\begin{aligned} C^i(s) &\mapsto X_s^I \\ C^i(s) &\mapsto \tilde{X}_s^I \end{aligned} \tag{2.4}$$

but this seems to be very difficult, by making an exact computation. So instead we consider a wavy line in Monge gauge

$$C^i(s) = (s, \xi^a(s)) \tag{2.5}$$

where $a = 1, 2, 3$, and compute the inverse just to lowest order in the fluctuations about a straight line. We get

$$\sigma_s^{1a} = s^2 \frac{d}{ds}(s^{-1}\xi^a(s)) \tag{2.6}$$

and from this, we can obtain $\xi^a(s)$ in terms of X_s^I by means of an integration. Then we get

$$\begin{aligned} \frac{\partial C^i(s)}{\partial X_t^I} &= -\frac{s}{t^2}\theta(s-t)\lambda_{1i}^I + \mathcal{O}(\xi) \\ \frac{\partial C^i(s)}{\partial \tilde{X}_t^I} &= -\frac{s}{t^2}\theta(s-t)\tilde{\lambda}_{1i}^I + \mathcal{O}(\xi). \end{aligned} \tag{2.7}$$

Here $\theta(\bullet)$ denotes the Heaviside step function. We can now check that these constitute the inverse mappings to

$$\begin{aligned} \frac{\partial X_t^I}{\partial C^i(u)} &= \lambda_{1i}^I t^2 \frac{d}{dt}\left(t^{-1}\delta(t-u)\right) + \mathcal{O}(\xi) \\ \frac{\partial X_t^I}{\partial \tilde{C}^i(u)} &= \tilde{\lambda}_{1i}^I t^2 \frac{d}{dt}\left(t^{-1}\delta(t-u)\right) + \mathcal{O}(\xi) \end{aligned} \tag{2.8}$$

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[4]For the zero mode part this is the Hopf map from S^3 to S^2.

in the sense that

$$\int dt \frac{\partial C^i(s)}{\partial X_t^I} \frac{\partial X_t^I}{\partial C^j(u)} = \delta_{ij}\delta(s-u)$$

$$\int dt \frac{\partial C^i(s)}{\partial \tilde{X}_t^I} \frac{\partial \tilde{X}_t^I}{\partial C^j(u)} = \delta_{ij}\delta(s-u),$$

$$\int ds \frac{\partial C^i(s)}{\partial X_t^I} \frac{\partial X_r^J}{\partial C^i(s)} = \delta_{IJ}\delta(t-r) \tag{2.9}$$

We may also note that

$$\frac{\partial X_s^I}{\partial \tilde{X}_t^J} = \int dr \frac{\partial X_s^I}{\partial C^i(r)} \frac{\partial C^i(r)}{\partial \tilde{X}_t^J} \sim (\lambda^I \tilde{\lambda}^J)_{11} \tag{2.10}$$

is a non-singular matrix. Hence $X \mapsto \tilde{X}$ is a well-behaved coordinate transformation.

We have seen that it is always possible to get $C^i(s)$ from σ_s^{ij} (in fact from σ_s^{i1} in Monge gauge) by means of an integration. We have also seen that X_s^I and $\tilde{X}_s^I$ are dependent coordinates. We may probably always choose either X_s^I or $\tilde{X}_s^I$ as independent coordinates, in place of (gauge fixed) coordinates $C^i(s)$. If we impose a gauge fixing on the parametrization of $C^i(s)$, then we have three independent coordinates that describes the loop. But such a gauge fixing on $C^i(s)$ does not imply any gauge fixing on X_s^I. So now we do have a matching number of coordinates.

3. The Bogomolnyi equation

In the abelian case, the Bogomolnyi equation for selfdual strings was obtained in [1]. It is given by

$$H_{ijk} = \epsilon_{ijlk}\partial_l \phi \tag{3.1}$$

where $H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}$ is the gauge field strenght of the two-form gauge potential B_{ij}.

To generalize this to the non-abelian case, we first introduce abelian loop space fields,

$$A_{is} = B_{ij}(C(s))\dot{C}^j(s)$$
$$\phi_{is} = \dot{C}_i(s)\phi(C(s)) \tag{3.2}$$

and define the field strength $F_{is,jt} = \frac{\delta}{\delta C^i(s)}A_{jt} - \frac{\delta}{\delta C^j(t)}A_{is}$. We get

$$F_{is,jt} = H_{ijk}(C(s))\dot{C}^j(s)\delta(s-t). \tag{3.3}$$

We then find that the Bogomolnyi equation can be written as

$$F_{is,jt} = \epsilon_{ijkl}\partial_{ks}\phi_{lt} \tag{3.4}$$

where $\partial_{is} := \frac{\delta}{\delta C^i(s)}$.

We now propose that the non-abelian generalization of this equation is given by [6]

$$F_{is,jt} = \epsilon_{ijkl} D_{ks} \phi_{lt}. \tag{3.5}$$

where $D_{is} = \partial_{is} + A_{is}$ is the gauge covariant derivative.

We can write the one-form gauge potential in loop space

$$\mathcal{A}(C) = \int ds A_{is} \delta C^i(s) \tag{3.6}$$

in the various coordinate systems as follows,

$$\begin{aligned} A(C) &= \int ds A_I(s, X) \delta X_s^I \\ &= \int ds \tilde{A}_I(s, \tilde{X}) \delta \tilde{X}_s^I \end{aligned} \tag{3.7}$$

From this, we get that

$$A_{is} = \lambda_{ij}^I \left(A_I(s) \dot{C}^j(s) + \frac{1}{2} \dot{A}_I(s) C^j(s) \right) \tag{3.8}$$

In concordance with this, we also let

$$\phi_{is}(C) = \dot{C}_i(s) \phi(s, X) + \frac{1}{2} C_i(s) \dot{\phi}(s, X) \tag{3.9}$$

for the scalar field.

In [6] it was shown that the SU(2) covariant Bogomolnyi equation

$$\frac{1}{2} \epsilon_{IJK} F_{IJ}(s, t) = D_K(s) \phi(t), \tag{3.10}$$

with the above definitions of the fields, implies the SO(4) covariant Bogomolnyi equation, eq. (3.5), for selfdual strings. Here $D_K(s) = \frac{\delta}{\delta X_s^K} + A_K(s)$. This however, does not show that any solution to eq. (3.5) can be obtained solely from eq. (3.10), and in fact this is not true. We need another copy of the SU(2) equation.

We can choose to express a field in terms of the coordinates X or the coordinates $\tilde{X}$. Let now $A = A(X)$ and $\tilde{A} = \tilde{A}(\tilde{X})$ commute, $[A, \tilde{A}] = 0$, and similarly $\phi = \phi(X)$ and $\tilde{\phi} = \tilde{\phi}(\tilde{X})$ also commute, $[\phi, \tilde{\phi}] = 0$. Then we may consider the two Bogomolnyi equations for these fields,

$$\begin{aligned} F_{IJ} &= \epsilon_{IJK} D_K \phi \\ \tilde{F}_{IJ} &= \epsilon_{IJK} \tilde{D}_K \tilde{\phi} \end{aligned} \tag{3.11}$$

and find that

$$\begin{aligned} A_{is} &= \partial_{is} X_t^I A_I + \partial_{is} \tilde{X}^{\tilde{I}} \tilde{A}_{\tilde{I}} \\ \phi_{is} &= \dot{C}_i(s) \left(\phi + \tilde{\phi} \right) + \frac{1}{2} C_i(s) \left(\dot{\phi} + \dot{\tilde{\phi}} \right) \end{aligned} \tag{3.12}$$

satisfies the Bogomolnyi equation.

Connection with local physics is provided by letting

$$B_{ij}(s) = \lambda_{ij}^I A_I(s) + \lambda_{ij}^{\tilde{I}} A_{\tilde{I}}(s) \tag{3.13}$$

for the two-form gauge potential. This relation can now be inverted, to express A_I and $\tilde{A}_I$ in terms of B_{ij} or in terms of A_{is}.

In the abelian case, we can then let $B_{ij}(s, C) = B_{ij}(C(s))$ be the usual local two-form gauge potential. For point-like loops our definition coincides with the conventional definition

$$A_{is} = B_{ij}(C(s))\dot{C}^j(s). \tag{3.14}$$

We now show the the SO(4) Bogomolnyi equation implies these two SU(2) Bogomolnyi equations. We compute

$$
\begin{aligned}
\epsilon_{IJK} F_{IJ}(s,t) &= \epsilon_{IJK} \int du \int dv \frac{uv}{s^2 t^2} \lambda_{1i}^I \lambda_{1j}^J \theta(u-s)\theta(v-t) F_{iu,jv} \\
&= \frac{1}{s^2 t^2} \epsilon_{IJK} \lambda_{1i}^I \lambda_{1j}^J \epsilon_{ijkl} \int du \int dv uv \theta(u-s)\theta(v-t) D_{ku}\phi_{lv} \\
&= 2\lambda_{1i}^K \frac{1}{s^2 t^2} \int du \int dv uv \theta(u-s)\theta(v-t) D_{iu}\phi_{1v}
\end{aligned}
\tag{3.15}
$$

We note that

$$\phi_{1s} = \phi(s) + \frac{1}{2} s\dot{\phi}(s) \tag{3.16}$$

in Monge gauge. Then we use

$$D_{iu} = \int dw \frac{\partial X_w^I}{\partial C^i(u)} D_I(w) \tag{3.17}$$

and get

$$
\begin{aligned}
\epsilon_{IJK} F_{IJ}(s,t) &= 2\lambda_{1i}^K \lambda_{i1}^I \frac{1}{s^2 t^2} \int du \int dv \int dw uv \\
&\quad \theta(u-s)\theta(v-t) \left(\delta(w-u) + w\dot{\delta}(u-w) \right) \\
&\quad D_I(w) \left(\phi(v) + \frac{1}{2} v\dot{\phi}(v) \right) \\
&= 2 D_K(s)\phi(t)
\end{aligned}
\tag{3.18}
$$

The same type of computation can be done to show that

$$\epsilon_{IJK} \tilde{F}_{IJ}(s,t) = 2\tilde{D}_K(s)\tilde{\phi}(t). \tag{3.19}$$

3.1 Constructing selfdual string solutions

In the abelian case, far away from a selfdual string, the U(1) field strength is of the form

$$H_{ijk}(x) = \epsilon_{ijkl} \frac{x^l}{|x|^4}. \tag{3.20}$$

The string being self-dual means that $H_{i05} \sim \epsilon_{ijkl}H_{jkl}$. From now on we will only focus on the components H_{ijk} of the field strength, which means that we may forget about self-duality. From eqs. (3.20), (3.3) we then get

$$F_{is,jt} = \epsilon_{ijkl}\frac{\sigma_s^{kl}}{(R_s)^4}\delta(s-t) \tag{3.21}$$

where $R_s = |C^i(s)|$. We would now like to transform this solution into our new coordinates in loop space. However, we do not get any nice (i.e. symmetric) expression if we just use the X coordinates. We can get a much nicer expression if we use both X and $\tilde{X}$ coordinates. We then first separate $F_{is,jt}$ into selfdual and antiself dual pieces[5] as

$$F_{is,jt} = F_{is,jt}^+ + F_{is,jt}^-. \tag{3.22}$$

We then apply the wavy line approximation. Now to be slightly more general, and to get s dimensionless, we introduce parameters R^a of dimension length, and parametrize the wavy line as

$$C^i(s) = (Rs, R\xi^a(s) + R^a) \tag{3.23}$$

where $R = \sqrt{R^a R_a}$ may be thought of as a radius. Then we find that

$$X_s^I X_s^I = R^4(1 + \mathcal{O}(s,\xi)) \tag{3.24}$$

Then noting the identity $(\lambda^I \lambda^J \lambda^K)_{11} = i\epsilon_{IJK}$, we get

$$F_{IJ}(s,t,X) = \frac{1}{R^2}\epsilon_{IJK}\int du \frac{u^2}{s^2t^2}\theta(u-s)\theta(u-t)\frac{X_u^K}{(R_u)^4} \tag{3.25}$$

from the F^+ piece, and similarly,

$$\tilde{F}_{IJ}(s,t,\tilde{X}) = \frac{1}{R^2}\epsilon_{IJK}\int du \frac{u^2}{s^2t^2}\theta(u-s)\theta(u-t)\frac{\tilde{X}_u^K}{(R_u)^4} \tag{3.26}$$

from F^-. These solutions look rather strange, but if we write $s^{-2} = -\partial_s(s^{-1})$, and make 'an integration by parts' by moving the derivative ∂_s to the theta function (eventhough there is no integration over s), and then do the same thing for t^{-2}, then we get

$$\begin{aligned} F_{IJ}(s,t,X) &= \frac{1}{R^2}\epsilon_{IJK}\frac{X_s^K}{(R_s)^4}\delta(s-t) \\ &= \epsilon_{IJK}\frac{X_s^K}{|X_s|^3}\delta(s-t)(1 + \mathcal{O}(\xi)) \end{aligned} \tag{3.27}$$

which equals the field strength from a Dirac monopole (in a way that is consistent with the accuracy of our approximation).[6]

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[5]By selfdual we simply mean with respect to ϵ_{ijkl}, and not with respect to some strange loop space hodgeduality operator. It should be noted that this decomposition into self-dual pieces has nothing to do with self-duality of $H_{\mu\nu\rho}$!

[6]Now this is up to a total derivative. But it seems likely that this total derivative term could appear in the $\mathcal{O}(\xi)$ as well.

If we would not use both X and $\tilde{X}$ coordinates, we would not get a solution that would resemble a Dirac monopole at all. This is why we could not get any selfdual string solution by solving just one SO(3) covariant Bogomolnyi equation — we need *two* copies of this equation. In general we need to solve these two decoupled copies of this equation and then look for the physical string solution as a certain linear combination of these solutions.

The simplest non-abelian solution to the SO(3) Bogomolnyi equation is

$$\begin{aligned} \phi(s,X) &= \varphi(X_s) \\ A_\mu(s,X) &= \mathcal{A}_\mu(X_s) \end{aligned} \tag{3.28}$$

where $\varphi(x) \sim 2\coth(2x) - \frac{1}{x}$ is the celebrated Higgs profile of the 't Hooft-Polyakov monopole, and $\mathcal{A}_\mu(x)$ may be taken as the hedgehog solution for the gauge field.

This solution can be obtained by imitating the ADHMN construction. We then take the trivial solution $T^I(s) = 0$ to the Nahm equation (to be presented shortly). Then the construction equation (also to be presented shortly) reads

$$\left(-\frac{\delta}{\delta z(t)} + X_t^I \sigma_I \right) v = 0 \tag{3.29}$$

where $z(s) \in [-1,1]$. This equation has solutions

$$v = N(X) P \exp\left(\int dt\, z(t) X_t^I \sigma_I \right) n \tag{3.30}$$

where n is an element in an orthonormal basis of vectors. P denotes a path ordered exponent. We separate these solutions into factors as

$$v = \prod_s v_s \tag{3.31}$$

This decomposition becomes unique if we in addition impose the normalization conditions

$$\int_{-1}^{1} dz(s)(v_s)^2 = 1 \tag{3.32}$$

for each s. We then find that

$$v_s = N(s,X) \exp\left(z(s) X_s^I \sigma_I \right) n \tag{3.33}$$

The normalization conditions yield

$$N(s,X) = \sqrt{\frac{|X_s|}{\sinh(2|X_s|)}} \tag{3.34}$$

(Here $|\bullet|$ denotes the standard euclidean norm.) and then we get

$$\begin{aligned} \phi(s,X) &= \prod_{r \neq s} \int dz(r)(v_r)^2 \int dz(s) z(s) (v_s)^2 \\ &= N(X_s)^2 \int_{-1}^{1} dz(s) z(s) \exp\left(2z(s) X_s^I \sigma_I \right). \end{aligned} \tag{3.35}$$

The evaluation of this integral gives the 't Hooft-Polyakov solution presented above.

In an attempt to find the general solution to one of these SO(3) Bogomolnyi equations, in the case of SU(2) gauge group, we use the ADHMN construction for magnetic monopoles. We thus make the following ansatz

$$A_I(s) = -i \int [dz] \prod_t v_t^\dagger \partial_I(s) v_t$$
$$\phi(s) = \int [dz] \prod_t z(s) v_t^\dagger v_t \qquad (3.36)$$

and we make a corresponding ansatz for the tilde fields $\tilde{A}$ and $\tilde{\phi}$. Here $v_t = v_t(z, X)$ and $z = z(s)$ is a one-dimensional loop. In the path integral we integrate over the range $z(s) \in [-1, 1]$ for each s. We normalize v_r as

$$\int_{-1}^1 dz(r) v_r^\dagger v_r = 1 \qquad (3.37)$$

for each r, and we let v_r be subject to the construction equations

$$\Delta_s^\dagger v = 0, \qquad (3.38)$$

where

$$\Delta_r(z) = \frac{\delta}{\delta z(r)} + \left(X_r^I - T^I(z, r)\right) \sigma^I \qquad (3.39)$$

and where the T^I obey the generalized Nahm equation

$$\frac{\delta T^K(s)}{\delta z(r)} + \frac{i}{2} \epsilon_{IJK} [T^I(s), T^J(r)] = 0. \qquad (3.40)$$

We then define v_r by eqs. (3.31), (3.32).

To see that this really gives solutions to our Bogomolnyi equation, one can just make very small modification of existing derivations of the ADHMN construction. We present such a slightly modified derivation in the appendix B.

Also one should specify the boundary conditions. By suitable choice of units (i.e. of the Higgs vev), we have already assumed that the range is $z(s) \in [-1, 1]$ and we now give the boundary conditions as

$$T^I(s) = -\frac{t^I(s)}{z(s) \mp 1} + \mathcal{O}(1). \qquad (3.41)$$

near $z(s) = \pm 1$. Then

$$2it^K(s)\delta(s-r) = \epsilon_{IJK}[t^I(s), t^J(r)] \qquad (3.42)$$

From this we conclude that

$$[t^I(s), t^J(r)] = \epsilon_{IJK}\delta(s-r)t^K(s) + K\delta^{IJ}\dot{\delta}(s-r) \qquad (3.43)$$

where we can allow for a central extension.

4. Nahm's equations from the membrane

We now wish to see if we can obtain the Nahm equation as the Bogomolnyi equation for the $M2$ brane. If the $M5$ brane is extended in the directions $0, 1, 2, 3, 4, 5$ and the M2 brane in the directions $0, 1, 6$ then the unbroken supersymmetries satisfy [5]

$$\Gamma^6 \epsilon = \Gamma^{2345} \epsilon \tag{4.1}$$

This brane configuration is translationally symmetric in all directions but the 6-direction. In particular the branes intersect along a string aligned along the 1-direction. We assume that only the scalar fields on M2 are excited which are parallel to M5 brane, and we denote these excited scalar fields as X^i for $i = 2, 3, 4, 5$. The supersymmetry variation for the fermions on the $M2$-brane (assuming the conjectured membrane theory in [4]) is given by

$$\delta\psi = -\Gamma^6 \Gamma_i \epsilon D_6 X^i - \frac{1}{6} \Gamma_{ijk} \epsilon [X^i, X^j, X^k]_{M2} \tag{4.2}$$

The condition for unbroken supersymmetry can be rewritten as

$$\epsilon^{ijkl} \Gamma_l \Gamma^6 \epsilon = \Gamma^{ijk} \epsilon \tag{4.3}$$

By demanding $\delta\psi = 0$ for such supersymmetry parameters, we find the Nahm equation

$$\partial_6 X^i \sim \epsilon_{ijkl} [X^j, X^k, X^l]_{M2}. \tag{4.4}$$

It seems like there is only one way one can extend the membrane theory if one wants to leave the SO(4) gauge group. This is to let the gauge group be of the form $\hat{SU}(N)$, where $\hat{SU}(N)$ denotes the loop algebra of SU(N) with a non-trivial central extension. More generally one could take the loop algebra extension of any semi-simple Lie algebra. We denote the generators as $T^a(s), 1$, where $\text{Tr}(T^a(s)) = 0$, and we choose a normalization such that $\text{Tr}(1) = 1$. We define the membrane three-bracket as [9]

$$[a, b, c]_{M2} = [a, b]\text{Tr}(c) + [b, c]\text{Tr}(a) + [c, a]\text{Tr}(b). \tag{4.5}$$

One may verify that this three-bracket satisfies the Fundamental Identity (the analog of the Jacobi identity for three-algebras, see [4]).

Let t^I generate an $su(2)$ subalgebra of $\hat{su}(N)$. Using the membrane bracket, we then find that

$$[t^I, t^J, 1]_{M2} = \epsilon^{IJK} t^K \tag{4.6}$$

and all the other brackets vanish, so for instance $[t^I, t^J, t^K]_{M2} = 0$.

If we take a gauge group that has $su(2) \oplus su(2)$ as a subalgebra, we can solve the Nahm equation by taking

$$\begin{aligned} X^I(z) &= T^I(z) + \tilde{T}^I(z) \\ X^4(z) &= 1 \end{aligned} \tag{4.7}$$

where x^6 is here denoted by z as conventional, thereby descending to the Nahm equation

$$\begin{aligned}
\partial_6 T^I &= \epsilon^{IJK} T^J T^K \\
\partial_6 \tilde{T}^I &= \epsilon^{IJK} \tilde{T}^J \tilde{T}^K.
\end{aligned} \tag{4.8}$$

This is the zero mode part of the Nahm equation that we found earlier in our construction of selfdual strings.

To get all modes from the membrane, we drop the integration over s, and let the fields be (non-abelian) loops themselves, $X^I(s) = X^I_a(s)T^a(s) + X^I_\sharp(s)\mathbf{1}$, $A_\mu(s) = A_{\mu,a}(s)T^a(s) + A_{\mu\sharp}(s)\mathbf{1}$ (here we associate the index $\sharp$ to the central element). This in turn implies a covariant derivative $D_\mu(s)$ meaning that the fields must be functionals of loops $x^\mu(s)$. We assume a local dependence on these loops, as $X^I(s,[x]) = X(x(s))$, $A_\mu(s,[x]) = A_\mu(x(s))$. Then we take the supersymmetry variation as

$$\begin{aligned}
\delta(t) X^I(s) &= i\delta(s-t)\bar{\epsilon}\Gamma^I\psi(s) \\
\delta(t) A_\mu(s) &= \imath\bar{\epsilon}\Gamma_\mu\Gamma_I[\psi(t), X^I(s), \bullet] \\
\delta(t)\psi(s) &= -\Gamma^\mu\Gamma_I\epsilon D_\mu(t)X^I(s) - \frac{1}{6}\Gamma_{IJK}\epsilon[X^I, X^J(t), X^K(s)].
\end{aligned} \tag{4.9}$$

The loop space supersymmetry algebra reads

$$[\delta_\epsilon(s), \delta_\eta(t)] = \delta(s-t)\bar{\epsilon}\Gamma^\mu\eta\partial_\mu(s) \tag{4.10}$$

and the supersymmetry variations close on the fermionic equation of motion

$$\Gamma^\mu D_\mu(s)\psi(t) + \frac{1}{2}\Gamma_{IJ}[\psi(s), X^I(t), X^J] = 0. \tag{4.11}$$

4.1 Reduction to Yang-Mills

It is now straightforward to reduce the membrane theory to super Yang-Mills theory. The first step in the reduction is to restrict to zero mode parts. Next we let just one scalar field (let us choose it to be $X^{(8)}$, or in an eleven dimensional notation, the eights scalar field would be denoted as $X^{(10)}$) have a non-trivial central element R. We thus let

$$X^{(8)} = R + X^{(8)}_a T^a \tag{4.12}$$

and we let the other scalars be of the form

$$X^M = X^M_a T^a \tag{4.13}$$

with no central element, for $I = M = 1, \ldots, 7$. Then the three-bracket reduces to a commutator,

$$[X^M, X^N, X^8] = R[X^M, X^N]. \tag{4.14}$$

The gauge field variation reduces to

$$\delta A_\mu = \bar{\epsilon}\Gamma_\mu\Gamma_{(10)}\psi \tag{4.15}$$

Finally

$$\delta\psi = \Gamma^\mu \Gamma_I \epsilon D_\mu X^I + \ldots \tag{4.16}$$

reduces to

$$\delta\psi = \frac{1}{2}\Gamma^{\mu\nu}\Gamma_{(10)}\epsilon F_{\mu\nu} + \Gamma^\mu \Gamma_M \epsilon D_\mu X^M + \ldots \tag{4.17}$$

by dualizing the eight's scalar according to

$$D_\mu X^{(8)} = \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}. \tag{4.18}$$

We thus take eqs. (4.18) and (4.12) as a defining equation for the field $X^{(8)}$ that makes the membrane theory reduce to Yang-Mills theory.

As motivation for this prescription we note that only one scalar field can be excited in the direction (that we labeled by coordinate $I = 8$) of the compact M-theory circle, and that this compact dimension corresponds to a commuting coordinate, hence to an element in U(1).

We note that supersymmetry enforces the M2 gauge field to contain no propagating degrees of freedom. When we reduce to D2, the gauge field somehow must acquire one degree of freedom. If we adopt the triple product in eq. (4.5) for the M2 theory, it seems impossible to write a supersymmetric action because with this triple product we can not fulfil the invariance property $\mathrm{Tr}([x,y,a]b) + \mathrm{Tr}(a[x,y,b]) = 0$ of the trace form which is needed for a supersymmetric action [4]. Still supersymmetry variations close on those equations of motion found in [4], but it seems there is no action from which they can be derived if we use this triple product.

5. Other options?

Given that the membrane bracket is the one we have specified in eq. (4.5), it is clear that the fields should take values in a centrally extended Lie algebra, since otherwise we do not get any non-vanishing traces. Could we get a non-trivial theory by taking a trivial central extension of the type $\mathrm{SU}(N) \times \mathrm{U}(1)$? If the central extension is trivial, then the U(1) fields will obey free equations of motion (since the central element commutes with anything). In this sense the trivially centrally extendend membrane theory is trivial, and is probably just isomorphic to Yang-Mills theory upon a field redefinition. To obtain a non-trivial interacting theory we must have a non-trivial central extension (given the assumed form of the three-bracket), and this means that we must go to an infinite-dimensional gauge group since there are no non-trivial central extensions of finitie Lie algebras. This then is another motivation why loop space seems to arise for the $M2$ theories — it seems to be the only way one can construct a non-trivial membrane theory.

It would certainly be nice if one could find a local field theory living on the membrane. Indeed there is one such local field theory, which corresponds to SO(4) gauge group. This theory is constructed using a different three-product, given by

$$[a,b,c] = G_5 abc \pm antisym. \tag{5.1}$$

and the scalar fields and fermions are then given as

$$X^I = X_i^I G^i$$
$$\psi = \psi_i G^i \tag{5.2}$$

Here G^i are coordinates on a fuzzy three sphere, that generalizes the four-dimensional gamma matrices γ^i, and G_5 is then the analog of γ_5.

It is not known if this theory can be reduced to Yang-Mills theory. It does not seem to be possible to generalize this SO(4) theory to any higher (or lower) rank gauge groups. The algebraic structure (the Fundamental Identity together with antisymmetric structure constants of the three-algebra) of the membrane theory does not seem to admit any interesting generalizations, as we show in appendix C.

At last, let us show the connection between our $SU(2) \times SU(2)$ approach and the existing SO(4) approach taken by Basu and Harvey in [7]. Our Nahm generators should be related to the (loop space generalization of the) Basu-Harvey generators T^i as

$$T^I(s) = \lambda_{ij}^I [T^i(s), T^j(s)]$$
$$t^I(s) = \lambda_{ij}^I [G^i(s), G^j(s)] \tag{5.3}$$

and analogously for the tilde generators. We find that

$$T^i(s) = \frac{G^i(s)}{\sqrt{z(s) \mp 1}} \tag{5.4}$$

satisfies the loop space generalization of the Basu-Harvey's equation

$$\frac{\delta T^i(s)}{\delta z(t)} \sim \epsilon_{ijkl}[T^j(s), T^k(t), T^l]. \tag{5.5}$$

provided that $G^i(s)$ satisfies the loop space fuzzy sphere equation

$$\delta(s-t)G^i(s) \sim \epsilon_{ijkl}[G^j(s), G^k(t), G^l]. \tag{5.6}$$

Here $T^i := \int ds T^i(s)$. It is a quite complicated story how the proportionality constant is determined, which depends on the choice of reducible representation of SO(4) via the integer n. This can be found in [7](v3).

Moreover we get

$$T^I(s) = \frac{t^I(s)}{z(s) \mp 1} \tag{5.7}$$

which satisfies the loop space Nahm equation

$$\frac{\delta T^I(s)}{\delta z(t)} = \epsilon_{IJK}[T^I(s), T^J(t)]. \tag{5.8}$$

since (by a slight extension of the short computation in footnote 1 in the Introduction),

$$4i\delta(s-t)t^I(s) = \epsilon_{IJK}[t^J(s), t^K(t)] \tag{5.9}$$

follows as a consequence of eqs. (5.6) and (5.3).

Let us finally show the general implication of the Basu-Harvey equation (for the zero mode part for simplicity) here. That is, we show that

$$\frac{dT^i}{dz} = \frac{1}{2(n+2)}\epsilon_{ijkl}G_5 T^j T^k T^l \tag{5.10}$$

implies

$$\frac{dT^I}{dz} = i\epsilon_{IJK}T^J T^K \tag{5.11}$$

where $T^I = \lambda_{ij}^I T^i T^j$. We compute the left-hand side,

$$
\begin{aligned}
\frac{dT^I}{dz} &= \lambda_{ij}^I \left(\frac{dT^i}{dz} T^j + T^i \frac{dT^j}{dz} \right) \\
&= \frac{1}{2(n+2)} \left(\lambda_{im}^I \epsilon_{ijkl} + \lambda_{ij}^I \epsilon_{imkl} \right) G_5 T^j T^k T^l T^m \\
&= \frac{1}{2(n+2)} i\epsilon_{IJK} \lambda_{ij}^J \lambda_{kl}^K \left(\mathcal{P}_+ - \mathcal{P}_- \right) T^i T^j T^k T^l \\
&= i\epsilon_{IJK} \lambda_{ij}^J \lambda_{kl}^K T^i T^j T^k T^l \\
&= i\epsilon_{IJK} T^J T^K
\end{aligned} \tag{5.12}
$$

In the second step we have assumed that $\{G_5, T^i\} = 0$. In the third step we use eq. (A.10) to rewrite the 't Hooft matrices. In fourth step we used $G_5 = \mathcal{P}_+ - \mathcal{P}_-$ and $\mathcal{P}_+ \lambda_{ij}^I G^{ij} = (n+3)t^I$, $\mathcal{P}_- \lambda_{ij}^I G^{ij} = -(n+1)t^I$. The definition of $\mathcal{P}_\pm$ is found in [2] (where it is denoted as $\mathcal{P}_{\mathcal{R}_\pm}$, reflecting the fact that this is a projector on the representations $\mathcal{R}_\pm$ of $so(4)$).

The same computation can be done for the tilde variables.

This shows that the Basu-Harvey equation implies the two Nahm equations. That means that the Basu-Harvey equation is identical to (or possibly a stronger equation than) two copies of the Nahm equation.

We finally note that what was given the interpretation as a mysterious coupling constant $\lambda \sim \frac{1}{2(n+2)}$ in the Basu-Harvey equation in Ref [7], now simply disappears when we reformulate this equation as two Nahm equations, and hence this coupling constant need no explanation as it is no longer there.

Acknowledgments

A. The isomorphism $su(2) \oplus su(2) \simeq so(4)$

We first embed SU(N) in SO($2N$) for any N. If t^I denote generators of SU(N), then the embedding in SO($2N$) is given as

$$t^I \mapsto \lambda^I := a_i^\dagger (t^I)^{ij} a_j \tag{A.1}$$

where

$$a_i = \frac{1}{2}\left(\gamma^{2i-1} - i\gamma^{2i}\right) \tag{A.2}$$

are annihilation operators constructed out of SO(2N) gamma matrices, which thus satisfy the algebra

$$\{a_i^\dagger, a_j\} = \delta_{ij} \tag{A.3}$$

of N creation and N annihilation operators.

Now this gives an embedding of SU(N) into the Dirac spinor representation of SO($2N$) because

$$a_i^\dagger a_j = \frac{1}{2}\delta_{ij} - \frac{i}{2}\left(M_{2i-1,2j-1} + M_{2i,2j}\right) - \frac{1}{2}\left(M_{2i-1,2j} - M_{2i,2j-1}\right) \tag{A.4}$$

The term δ_{ij} does not contribute to the embedding matrices because the SU(N) matrices are traceless. This result was now derived in the spinor representation where

$$M_{ij} = \frac{i}{2}\gamma_{ij}. \tag{A.5}$$

But we may now take the M_{jk} as generators of SO($2N$) in any representation. In particular we may take the defining vector representation.

Taking $N = 2$, we obtain the 't Hooft matrices ($I = 1, 2, 3$)

$$\lambda^I = \frac{1}{2}\epsilon^{IJK}M^{JK} - M^{I4}. \tag{A.6}$$

Of course SO(4) $\simeq$ SU(2) $\times$ SU(2), and we can embed another SU(2) as

$$\tilde{\lambda}^I = \frac{1}{2}\epsilon^{IJK}M^{JK} + M^{I4} \tag{A.7}$$

Conversely, the λ^I, $\tilde{\lambda}^I$ generate all of SO(4).

The 't Hooft matrices satisfy

$$\begin{aligned}
\lambda^I\lambda^J &= i\epsilon^{IJK}\lambda^K + \delta^{IJ} \\
\tilde{\lambda}^I\tilde{\lambda}^J &= i\epsilon^{IJK}\tilde{\lambda}^K + \delta^{IJ} \\
[\lambda^I, \tilde{\lambda}^J] &= 0
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
\lambda_{ij}^I\lambda_{kl}^I &= -2\delta_{ij,kl} - \epsilon_{ijkl} \\
\tilde{\lambda}_{ij}^I\tilde{\lambda}_{kl}^I &= -2\delta_{ij,kl} + \epsilon_{ijkl}
\end{aligned} \tag{A.9}$$

From these relations we also derive the identities

$$\begin{aligned}
\epsilon_{ijkm}\lambda_{ml}^K &= i\epsilon_{IJK}\lambda_{ij}^I\lambda_{kl}^J - \delta_{ik}\lambda_{jl}^K + \delta_{jk}\lambda_{il}^K - \delta_{lk}\lambda_{ij}^K \\
-\epsilon_{ijkm}\tilde{\lambda}_{ml}^K &= i\epsilon_{IJK}\tilde{\lambda}_{ij}^I\tilde{\lambda}_{kl}^J - \delta_{ik}\tilde{\lambda}_{jl}^K + \delta_{jk}\tilde{\lambda}_{il}^K - \delta_{lk}\tilde{\lambda}_{ij}^K
\end{aligned} \tag{A.10}$$

Also, we have

$$\begin{aligned}
\mathrm{Tr}(\lambda^I\lambda^J) &= 4\delta^{IJ} \\
\mathrm{Tr}(\tilde{\lambda}^I\lambda^J) &= 0.
\end{aligned} \tag{A.11}$$

B. Verifying the ADHMN construction in loop space

Here we check the steps in the verification of the Nahm construction, following closely the steps in [8]. We make the ansatz

$$A_I(s) = -i \int [dz] v_t^\dagger \partial_I(s) v_t$$

$$\phi(s) = \int [dz] z(s) v_t^\dagger v_t \tag{B.1}$$

We will suppress the multiplication sign over t. Here

$$\Delta_s^\dagger v_s = 0$$

$$\int [dz] v_r^\dagger v_r = 1 \tag{B.2}$$

and

$$\Delta_r(z) = \frac{\delta}{\delta z(r)} + \left(X_r^I - T^I(z, r) \right) \sigma^I, \tag{B.3}$$

$$\frac{\delta T^K(s)}{\delta z(r)} + \frac{i}{2} \epsilon_{IJK} [T^I(s), T^J(r)] = 0 \tag{B.4}$$

which is the generalized Nahm equation.

We verify that this solves the Bogomolnyi equation by first computing

$$\frac{1}{2} \epsilon_{IJK} F_{IJ}(s, t) = -i\epsilon_{IJK} \left(\int [dz] \partial_I(s) v_r^\dagger \partial_J(t) v_r + \int [dz][dz'] \underbrace{v_r^\dagger \partial_I(s) v_r}_{-(\partial_I(s) v_r^\dagger) v_r} v_{r'}^\dagger \partial_J(t) v_{r'} \right)$$

$$= -i\epsilon_{IJK} \int [dz][dz'] \partial_I(s) v_r^\dagger(z) F_{rr'}(z, z') \partial_J(t) v_{r'}(z') \tag{B.5}$$

Here

$$F_{rr'}(z, z') = \delta(z - z')\delta(r - r') - v_r(z) v_{r'}^\dagger(z') \tag{B.6}$$

obeys

$$\int [dz'] \prod_{r'} F_{rr'}(z, z') F_{r'r''}(z', z'') = F_{rr''}(z, z'') \tag{B.7}$$

(Here $\delta(z)$ is defined as $\int [dz]\delta(z)\phi(z) = \phi(0)$, and $\delta(r)$ is defined with respect to multiplication, hence it should perhaps more appropriately be written as $e^{\delta(r)}$.) To see this, the normalization condition (3.38) is an essential ingredient. Hence F is a projection operator onto the space orthogonal to the kernel of $\Delta^\dagger$, and can be written as

$$F_{rr'}(z, z') = \Delta_r(z) G_{r,r'}(z, z') \Delta_{r'}^\dagger(z') \tag{B.8}$$

– 18 –

where

$$G_{rs}(z, z') = (\delta(r - s)\delta(z - z')\Delta_r^\dagger(z)\Delta_s(z'))^{-1} \tag{B.9}$$

We can then continue on our line and get

$$\frac{1}{2}\epsilon_{IJK}F_{IJ}(s,t) = -i\epsilon_{IJK}\int [dz][dz']\underbrace{\partial_I(s)v_r^\dagger(z)\Delta_r(z)}_{\left(\Delta_r^\dagger(z)\partial_I(s)v_r(z)\right)^\dagger}G_{rr'}(z,z')\Delta_{r'}^\dagger(z')\partial_J(t)v_{r'}(z')$$

$$= -i\epsilon_{IJK}\int [dz][dz']v_r^\dagger(z)\sigma_I\delta(s-r)G_{rr'}(z,z')\sigma_J\delta(r'-t)v_{r'}(z')$$

Finally we note that $G_{rr'}(z, z')$ is diagonal when the Nahm equation is obeyed, hence we can commute σ_I with G and use $[\sigma_I, \sigma_J] = 2i\epsilon_{IJK}\sigma_K$ to get

$$= 2\int [dz][dz']v_s^\dagger(z)\sigma_K G_{st}(z,z')v_t(z') \tag{B.10}$$

Let us now turn to the right-hand side,

$$D_I(s)\phi(t) = \int [dz]z(t)D_I(s)(v_r^\dagger v_r)$$

$$= \int [dz][dz']\partial_I(s)v_r^\dagger(z)F_{rr'}(z,z')z'(t)v_{r'}(z')$$

$$+ \int [dz][dz']z(t)v_r^\dagger(z)F_{rr'}(z,z')\partial_I(s)v_{r'}(z') \tag{B.11}$$

Then we again replace $F_{rr'}(z, z') = \Delta_r(z)G_{rs}(z, z')\Delta_r^\dagger(z')$ and use

$$\begin{aligned}\Delta_r^\dagger(z)v_r(z) &= 0\\ \Delta_r^\dagger(z)z(t) &= \delta(t - r)\\ \Delta_r^\dagger(z)\partial_I(s)v_r(z) &= -\delta(s - r)\sigma_I v_r(z)\end{aligned} \tag{B.12}$$

and get immediately that

$$D_I(s)\phi(t) = 2\int [dz][dz']v_s^\dagger(z)\sigma_I G_{st}(z,z')v_t(z') \tag{B.13}$$

This concludes the verification that we have indeed constructed solutions to the Bogomolnyi equation

$$F_{IJ}(s,t) = \epsilon_{IJK}D_K(s)\phi(t). \tag{B.14}$$

C. No-go argument

In this appendix we make it plausible that there are no non-trivial finite-dimensional extensions of the SO(4) membrane theory within the class of semi-simple Lie algebras, like SU(N) if we require a totally antisymmetric structure constants f^{abcd} of the three-algebra, as defined below.

C.1 The associated Lie algebra

Assume the T^a generate a three-algebra

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d \tag{C.1}$$

where the three-bracket is subject to the Fundamental Identity [4]

$$[ab[cde]] = [[abc]de] + [c[abd]e] + [cd[abe]] \tag{C.2}$$

By just assuming the antisymmetry property $[abc] = -[bac]$, we can write this in the following equivalent form

$$[ab[cde]] - [cd[abe]] = [[abc]de] - [[abd]ce] \tag{C.3}$$

which reflects the Lie algebra structure of the three-algebra. More clearly, if we define a linear operator as

$$t^{ab}(\bullet) = [T^a, T^b, \bullet] \tag{C.4}$$

then the three-algebra says that

$$t^{ab}(T^c) = f^{abc}{}_d T^d \tag{C.5}$$

and then

$$\begin{aligned}
[t^{ab}, t^{cd}](T^e) &= t^{ab}(f^{cde}{}_f T^f) - t^{cd}(f^{abe}{}_f T^f) \\
&= \left(f^{abf}{}_g f^{cde}{}_f - f^{cdf}{}_g f^{abe}{}_f \right) T^g
\end{aligned} \tag{C.6}$$

Applying the fundamental identity, we get

$$\begin{aligned}
&= \left(f^{abc}{}_g f^{gde}{}_f - f^{abd}{}_g f^{gce}{}_f \right) T^f \\
&= -2 f^{ab[c}{}_g t^{d]g}(T^e)
\end{aligned} \tag{C.7}$$

The commutator is again a linear operator, and multiplication being defined by composition of linear operators, is associative. Hence the commutator satisfies the Jacobi identity.

Not all the t^{ab} need to be linearly independent linear operators. Let us define

$$t^A := \Gamma^A_{ab} t^{ab} \tag{C.8}$$

in such a way that the t^A become linearly independent and is the maximal such set. Then the commutator must close into this set by the above result. That is,

$$[t^A, t^B] = C^{AB}{}_C t^C \tag{C.9}$$

C.2 More structure

Let us define

$$t^{ab} := \Gamma_A^{ab} t^A \tag{C.10}$$

where t^A is the maximal set of linearly independent such operators (the operators $[T^a, T^b, *]$ need not be linearly independent). That means that Γ_A^{ab} viewed as a matrix with row-index A and column index ab, need not be a square matrix. If so, then we can just find a one-sided inverse,

$$\Gamma_A^{ab} \Gamma_{ab}^B = \delta_A^B \tag{C.11}$$

that is, if the index range A is less than the index range ab. Using this, we may thus invert the above definition and get

$$t^A = \Gamma_{ab}^A t^{ab} \tag{C.12}$$

We now also define

$$t^{A,b}{}_e := \Gamma_{cd}^A f^{cdb}{}_e \tag{C.13}$$

and hence

$$t^A(T^b) = t^{A,b}{}_e T^e \tag{C.14}$$

Contracting the Fundamental Identity

$$[t^{ef}(T^a), T^b, \bullet] - [t^{ef}(T^b), T^a, \bullet] = [t^{ef}, t^{ab}](\bullet) \tag{C.15}$$

by Γ_{ef}^A, we get

$$[t^A(T^a), T^b, \bullet] - [t^A(T^b), T^a, \bullet] = [t^A, t^{ab}] \tag{C.16}$$

or

$$t^{Aa}{}_c t^{cb} - t^{Ab}{}_c t^{ca} = [t^A, t^{ab}] \tag{C.17}$$

or

$$\left(t^{Aa}{}_c \Gamma_C^{cb} - t^{Ab}{}_c \Gamma_C^{ca} \right) t^C = \Gamma_C^{ab} [t^A, t^C] \tag{C.18}$$

Now since t^A are linearly independent, and generate a semisimple Lie algebra, they must be associated with a non-degenerated Killing form

$$\kappa^{AB} = \text{Tr}(t^A t^B) \tag{C.19}$$

which we can use to conclude from the above that

$$[t^A, \Gamma^B] = C^{AB}{}_C \Gamma^C \tag{C.20}$$

– 21 –

Conversely one may check that

$$f^{abc}{}_d = \Gamma^{ab}{}_A t^{Ac}{}_d. \tag{C.21}$$

satisfies the Fundamental Identity using the two equations

$$\begin{aligned}
[t^A, t^B] &= C^{AB}{}_C t^C \\
[t^A, \Gamma^B] &= C^{AB}{}_C \Gamma^C
\end{aligned} \tag{C.22}$$

This means that instead of trying to solve the Fundamental Identity, we may instead solve the two equivalent equations above.

An obvious solution is to take

$$\Gamma^{Aab} = t^{Aab} \tag{C.23}$$

In essence it is the only solution. To see this we make the ansatz

$$\Gamma_A = X_{AB} t^B \tag{C.24}$$

Then we find that

$$C^{AB'C} X^B{}_{B'} = C^{AB}{}_{C'} X^{C'}{}_C \tag{C.25}$$

Since we are interested in

$$f^{abcd} = \Gamma^{Aab} t^{Acd} = X_{AB} t^{Aab} t^{Bcd} \tag{C.26}$$

being completely antisymmetric, we may assume that $X^{AB} = X^{BA}$. Then the condition we find is

$$[X, t^A] = 0 \tag{C.27}$$

in the adjoint representation. Hence, by Schur's lemma, $X = 1$ for a simple Lie algebra where the adjoint is an irrep.

The adjoint representation is not irreducible for semisimple Lie algebras though, and in that case we can have $X = \lambda$ with different constants λ in different irreps, and we now have that

$$f^{abcd} = X_{AB} (t^A)^{ab} (t^B)^{cd}. \tag{C.28}$$

This is now the most general solution of the Fundamental Identity.

We now ask if this can become totally antisymmetric in $abcd$. It is manifestly so in ab and cd separately, as well as under exchange between the pairs. For complete antisymmetry in all indices, it thus suffices to find solutions such that

$$f^{abcd} + f^{acbd} = 0. \tag{C.29}$$

C.3 SU(2) ⊂ SO(4)

SU(2) can be embedded into SO(4) by taking

$$
\begin{aligned}
t^1 &= M^{23} + M^{14} \\
t^2 &= M^{31} + M^{24} \\
t^3 &= M^{12} + M^{34}
\end{aligned}
\tag{C.30}
$$

where

$$
(M^{ab})_{cd} = \delta^{ab}_{cd}
\tag{C.31}
$$

are the standard generators of SO(4) in an orthogonal basis, with respect to the invariant metric δ^{ab}.

We then compute

$$
\sum_{I=1}^{3} (t^I)^{ab}(t^I)^{cd} = \delta^{23}_{ab}\delta^{23}_{cd} + \delta^{23}_{ab}\delta^{14}_{cd} + \delta^{14}_{ab}\delta^{23}_{cd} + \delta^{14}_{ab}\delta^{14}_{cd} + \dots
\tag{C.32}
$$

For this particular case of SO(4) there is room for another SU(2),

$$
\begin{aligned}
\tilde{t}^1 &= M^{23} - M^{14} \\
\tilde{t}^2 &= M^{31} - M^{24} \\
\tilde{t}^3 &= M^{12} - M^{34}
\end{aligned}
\tag{C.33}
$$

and we get cancelation of terms like $\delta^{ab}_{12}\delta^{cd}_{12}$ and get

$$
\begin{aligned}
(t^I)^{ab}(t^I)^{cd} - (\tilde{t}^I)^{ab}(\tilde{t}^I)^{cd} &= \delta^{ab}_{23}\delta^{cd}_{14} + \delta^{ab}_{14}\delta^{cd}_{23} \\
&\quad + \delta^{ab}_{31}\delta^{cd}_{24} + \delta^{ab}_{24}\delta^{cd}_{31} \\
&\quad + \delta^{ab}_{12}\delta^{cd}_{34} + \delta^{ab}_{34}\delta^{cd}_{12} \\
&= \epsilon^{abcd}
\end{aligned}
\tag{C.34}
$$

that is, a completely antisymmetric tensor.

C.4 SU(N) ⊂ SO(M)

SU(N) can be embedded into the vector representation of SO(2N), but for $N > 2$ we can not fit any more semi-simple group into SO(2N). This means that we can not cancel the symmetric terms, like for instance $\delta^{ab}_{14}\delta^{cd}_{14}$. We may embed SU(N) × SU(N′) × ... in some SO(M) where $M > 2N$. But then there will always survive some symmetric terms that can not canceled. The analog of the 't Hooft matrices will sit as $2N \times 2N$ block matrices inside the SO(M) $M \times M$ matrices (in the vector representation), which for $N > 2$ will be strictly smaller than the $M \times M$ matrices. Therefore there will always be uncanceled symmetric terms that contributes to f^{abcd} when $N > 2$.

C.5 Weaker conditions

Could it be that one could relax the assumption that f^{abcd} be totally antisymmetric and still get a sensible membrane theory? If we just assume the symmetries

$$f^{abcd} = f^{[ab][cd]} = f^{[cd][ab]} \tag{C.35}$$

we could find many solutions to the Fundamental Identity. However if we assume a three-bracket

$$[a, b, c\} = -[b, a, c\} \tag{C.36}$$

with no other symmetries, then we must also assume that

$$[[a, b, c\}, \bullet, d\} - [d, \bullet, [a, b, c\}\} \pm antisym = 0 \tag{C.37}$$

in order to close SUSY up to gauge transformations (which must be defined using an antiosymmetrized three-bracket). But this condition leads to the condition that the anti-symmetric part of the bracket $[\bullet, \bullet, \bullet]$ satisfies the Fundamental Identity as well, and hence we are back at where we started, which is to find non-trivial solutions of the Fundamental Identity for the totally antisymmetric three-bracket.

We may instead relax the assumption that

$$(t^A)^{ab} \tag{C.38}$$

be antisymmetric. That is, we relax the assumptions that f^{abcd} is totally antisymmetric, and instead just assume the it is antisymmetric in its three first indices, $f^{[abc]d}$. This will not affect the supersymmetry variations of the membrane since these are introduced without any reference to the trace form $\text{Tr}(T^a T^b)$. Then the Fundamental Identity becomes equivalent with the equations

$$[t^A, t^B] = C^{AB}{}_C t^C$$
$$2(t^A \Gamma^B)^{[ab]} = C^{AB}{}_C \Gamma^{C,ab} \tag{C.39}$$

One infinite set of solutions to these equations is provided by letting $A = ef$ and then

$$t^{ef,c}{}_d = \delta^{[e}_\sharp F^{fc]}{}_d$$
$$\Gamma^{ab}_{ef} = \delta^{ab}_{cd} \tag{C.40}$$

where $F^{ab}{}_c$ are structure constants in any Lie algebra, thus satisfying the Jacobi identity.

The Lie algebra generators are $t^{a\sharp}$ together with the $\mathrm{U}(1)$ generators t^{ab}.

The three-bracket now becomes

$$[a, b, c] = [a, b]\text{Tr}(c) + [b, c]\text{Tr}(a) + [c, a]\text{Tr}(b). \tag{C.41}$$

This solution to the Fundamental Identity was found in [9].

References

[1] P.S. Howe, N.D. Lambert and P.C. West, *The self-dual string soliton*, *Nucl. Phys.* **B 515** (1998) 203 [hep-th/9709014].

[2] S. Ramgoolam, *On spherical harmonics for fuzzy spheres in diverse dimensions*, *Nucl. Phys.* **B 610** (2001) 461 [hep-th/0105006].

[3] D.S. Berman and N.B. Copland, *A note on the M2-M5 brane system and fuzzy spheres*, *Phys. Lett.* **B 639** (2006) 553 [hep-th/0605086].

[4] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955]; *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738];
A. Gustavsson, *Algebraic structures on parallel M2-branes*, arXiv:0709.1260.

[5] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108].

[6] A. Gustavsson, *Loop space, (2, 0) theory and solitonic strings*, *JHEP* **12** (2006) 066 [hep-th/0608141]; *Closed non-abelian strings*, arXiv:0705.3369.

[7] A. Basu and J.A. Harvey, *The M2-M5 brane system and a generalized Nahm's equation*, *Nucl. Phys.* **B 713** (2005) 136 [hep-th/0412310].

[8] E.J. Weinberg and P. Yi, *Magnetic monopole dynamics, supersymmetry and duality*, *Phys. Rept.* **438** (2007) 65 [hep-th/0609055].

[9] H. Awata, M. Li, D. Minic and T. Yoneya, *On the quantization of Nambu brackets*, *JHEP* **02** (2001) 013 [hep-th/9906248].

[10] A. Gustavsson, *The non-Abelian tensor multiplet in loop space*, *JHEP* **01** (2006) 165 [hep-th/0512341].

[11] F. Girelli and H. Pfeiffer, *Higher gauge theory — Differential versus integral formulation*, *J. Math. Phys.* **45** (2004) 3949 [hep-th/0309173].

[12] J.C. Baez and U. Schreiber, *Higher gauge theory*, math.DG/0511710.

[13] U. Schreiber, *From loop space mechanics to nonabelian strings*, hep-th/0509163.

[14] C. Hofman, *Nonabelian 2-forms*, hep-th/0207017.

[15] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658 [hep-th/9706109].